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Translated by E.L.S.

J. Appl. Maths Mechs., Vol 55, No. 3, pp. 325-332, 1991
Printed in Great Britain

0021-8928/91 $15.00+0.00

WAVE MOTIONS CAUSED BY A SOURCE IN A FLUID OF VARIABLE DEPTH*

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Solutions of the problem of the wave motion produced by a pulsed source moving in a fluid over an inclined bottom are obtained. An asymptotic analysis of the solution is carried out and the structures of the wave fields are investigated.

The motion of a source in a fluid of constant depth has been quite thoroughly studied by the successive application of integral transforms and the stationary phase method /1-3/. An asymptotic theory of wave motions has been developed /4/ for small variations of a base of arbitrary form. This is based on the use of the apparatus of pseudodifferential operators and the reduction of the problem to the solution of Hamiltonian systems. Only some special cases have been considered when there are significant changes in depth (the fluid is bounded by a planar inclined bottom): the problem has been formulated of the structure of the wave wake behind a moving source and a method of solving it has been pointed out in /5/, and a solution of the planar problem for a pulsating source has been constructed in /6/.

1. Let a source of intensity b, pulsating at a frequency ω and moving at a velocity c parallel to the shore line be placed in a fluid which occupies a wedge-shaped domain at the instant of time $t=0$ (Fig.1).

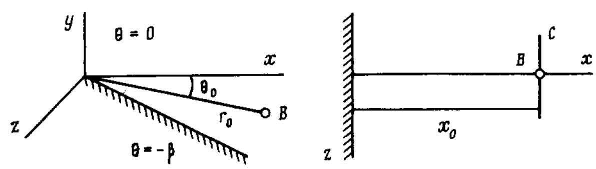

Fig.1

We shall write the equations, the boundary conditions and the initial conditions of the problem within the framework of linear dispersion theory /2, 7/

$$\Delta G = -b\,(4\pi r)^{-1}\,\delta\,(r - r_0, \theta - \theta_0, z)\,E\,(-\omega t) \tag{1.1}$$

$$G_{tt} + 2cG_{tz} + c^2G_{zz} + gr^{-1}G_\theta = 0,\quad \theta = 0$$

$$G_\theta = 0,\quad \theta = -\beta;\quad G = G_t = 0,\quad \theta = 0,\quad t = 0$$

$$G < \infty,\quad r \to 0;\quad G \to 0,\quad \sqrt{r^2 + z^2} \to \infty;\quad \eta = -g^{-1}G_t\,|_{\theta=0}$$

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2},\quad \beta = \frac{\pi}{2n},\quad n = 2m+1, m = 0,1,2,\ldots$$

$$E\,(\varkappa) = \exp\,(i\varkappa)$$

*Prikl.Matem.Mekhan.,55,3,401-409,1991

Here, G is a non-stationary Green's function, r, θ and z are cylindrical coordinates, $(r_0, \theta_0, 0)$ is the position of the source B, t is the time, g is the acceleration due to gravity and η is the elevation of the free surface.

Using the method of integral transforms, we obtain the solution of problem (1.1) in the form

$$G = G^c + G^d$$

where G^c and G^d are determined by the eigenfunctions of the continuous and discrete spectra respectively /8/.

On passing to the limit as $t \to \infty$, we write the formulae for the steady-state motion

$$G_s = \lim_{t\to\infty} G = E(-\omega t)(G_s^c + G_s^d) \tag{1.2}$$

$$G_s^c = \frac{b}{16\pi^2} \lim_{t\to\infty} \int_0^\infty \int_0^\pi E(s_c z \cos\lambda) \sqrt{gs_c} \left[\frac{E(\Omega_c^- t) - 1}{\Omega_c^-} - \frac{E(\Omega_c^+ t) - 1}{\Omega_c^+}\right] Y^c ds_c\, d\lambda$$

$$G_s^d = \frac{bg}{16\pi^2} \sum_{l=1}^{n-1} \varepsilon_l \lim_{t\to\infty} \int_0^\infty \frac{E(\sigma z)}{\sqrt{gp\gamma}} \left[\frac{E(\Omega_d^- t) - 1}{\Omega_d^-} - \frac{E(\Omega_d^+ t) - 1}{\Omega_d^+}\right] Y^d(p)\, p\, d\sigma \tag{1.3}$$

$$Y^c = \Phi^c(r_0, \theta_0)\, \Phi^c(r, \theta), \quad Y^d(\varkappa) = \Phi_l^d(\varkappa r_0, \theta_0)\, \Phi_l^d(\varkappa r, \theta)$$

$$\Omega_c^\pm = \omega \pm \sqrt{gs_c} - cs_c \cos\lambda, \quad \Omega_d^\pm = \omega \pm \sqrt{gp\gamma} - c\sigma$$

$$s_c = \sqrt{p^2 + q^2}, \quad \gamma = \cos l\beta, \quad p, q \geqslant 0, \quad \sigma = p \quad \text{при} \quad \sigma \geqslant 0$$

$$\varepsilon_l = (-1)^{(n+1)/2} \left\{2(1+\gamma)^{2(n-1)} \sqrt{1-\gamma^2} \prod_{k=1,\, k\neq l}^{n-1} \sin(k+l)\beta \sin(k-l)\beta\right\}^{-1}$$

where Φ^c and Φ_l^d are the eigenfunctions of the continuous and discrete spectra respectively, which have been determined in /8/:

$$\Phi^c = \frac{1}{2\sqrt{2\pi}} \sum_{j,\chi=0}^{1} \sum_{k=1}^{n} B_k^{(\chi)} \exp\Big\{-s_c r \cos(a_{kj} + (-1)^j \theta) +$$

$$(-1)^\chi i \left(-rq \cos(a_{kj} + (-1)^j \theta) + \frac{\pi}{4}(n-1)\right)\Big\}$$

$$a_{kj} = 2(k-j)\beta$$

$$B_k^{(\chi)} = \frac{(-1)^{n-k}}{(s_c + p)^{n-1}} \prod_{\sigma=1}^{n-1} (p^2 \sin^2 \sigma\beta + q^2)^{-1/4} \times$$

$$\prod_{\sigma=n-k+1}^{n-1} (p^2 \sin^2 \sigma\beta + q^2) \prod_{\sigma=1}^{n-k} \operatorname{ctg} \sigma\beta \left[\frac{1}{2} p^2 \sin 2\sigma\beta + (-1)^\chi iqs_c\right]$$

$$\Phi_l^d = \frac{1}{2\sqrt{2}} \sum_{j=0}^{1} \sum_{k=1}^{n-l} B_{kl} \exp(-rp \sin(a_{kj} + (-1)^j \theta + l\beta))$$

$$B_{kl} = (-1)^{n-k} \prod_{\sigma=n-k+1}^{n-1} \sin(\sigma - l)\beta \sin(\sigma + l)\beta \times$$

$$\prod_{\sigma=1}^{n-k} \operatorname{ctg} \sigma\beta \sin(\sigma + l)\beta \cos(\sigma - l)\beta$$

Let us now investigate the quantity G_s^c for the waves of the continuous spectrum. We shall carry out this investigation by analogy with the problems considered in /2, 3/ for the case of an infinitely deep fluid.

An analysis of the zeros of the expressions $\Omega_c^\pm$, carried out as in /7/, shows that, in order to ensure exponential decay when $t \to \infty$, the path in the complex plane s_c for the interval $0 \leqslant \lambda < \pi/2$ must be displaced below the real axis in the neighbourhood of the points

$$s_c^\pm = \left(\sqrt{g}\, \frac{1 \pm \sqrt{1 + 4\tau\cos\lambda}}{2c\cos\lambda}\right)^2, \quad \tau = \frac{c\omega}{g}$$

We shall denote this deformed path by C_1. For the interval

$$\pi/2 < \lambda \leqslant \pi - \gamma_0, \quad \gamma_0 = \begin{cases} 0, & \tau < 1/4 \\ \arccos[q(4c\omega)^{-1}], & \tau \geqslant 1/4 \end{cases}$$

the path in the complex plane s_c must be deformed in such a way that it lies below the real axis in the neighbourhood of the point s_c^- and above the real axis in the neighbourhood of the point s_c^+. We shall denote this path by C_2. It follows from this that

$$G_s^c = \frac{bg}{8\pi^2}\left\{\int\limits_0^{\pi/2}\int\limits_{C_1} + \int\limits_{\pi/2}^{\pi-\gamma_0}\int\limits_{C_2} + \int\limits_{\pi-\gamma_0}^{\pi}\int\limits_0^{\infty}\right\} Y^c \frac{E(s_c z\cos\lambda)}{gs_c - (\omega - cs_c\cos\lambda)^2}\, ds_c\, d\lambda \tag{1.4}$$

The paths C_1 and C_2 are shown in Fig.2.

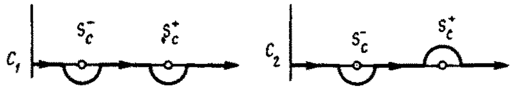

Fig.2

By carrying out an analogous investigation of G_s^d, we obtain from (1.3) an expression for the waves of the discrete spectrum

$$G_s^d = \frac{bg}{8\pi^2}\left\{\sum_{l=1}^{n-1}\varepsilon_l\int\limits_{L_1}\Psi_l^d(p)\,p\,dp + \sum_{l=1}^{l^*}\varepsilon_l\int\limits_{L_2}\Psi_l^d(\sigma)\,p\,d\sigma + \sum_{l=l^*+1}^{n-1}\int\limits_{-\infty}^{0}\Psi_l^d(\sigma)\,p\,d\sigma\right\} \tag{1.5}$$

$$\Psi_l^d(\varkappa) = \frac{E(\varkappa z)}{gp\gamma - (\omega - c\varkappa)^2}\,Y^d(p)$$

The deformed path L_1 in the p-plane coincides with the deformed path C_1 in the s_c-plane while the path L_2 differs from C_2 in that it is located on the negative real semi-axis of the σ-plane. The poles $p_{1,2}$ are determined using the formulae

$$p_{1,2} = g\,\frac{2\tau + \gamma \pm \sqrt{\gamma(\gamma + 4\tau)}}{2c^2} \tag{1.6}$$

and the poles $\sigma_{1,2}$ are determined using formulae which differ from (1.6) in that γ is replaced by $-\gamma$ and they exist subject to the condition $\gamma \geqslant 4\tau$, that is, for $l = l^* + 1, \ldots, n-1$, where $l^* = \beta^{-1}\arccos 4\gamma$.

Introducing the variable $\nu = \omega^2/g$, we get from formulae (1.4) and (1.5) by Cauchy's residue theorem

$$G_s^c = \frac{bi}{4\pi}\left\{\int\limits_0^{\pi/2}(I^- - I^+)\,d\lambda - \int\limits_{\pi/2}^{\pi-\gamma_0} I^+\,d\lambda\right\} \tag{1.7}$$

$$G_s^d = \frac{bi}{4\pi}\left\{\sum_{l=1}^{n-1}\varepsilon_l\sum_{\alpha=1}^{2}\frac{(-1)^\alpha p_\alpha E(p_\alpha z)}{\sqrt{\gamma(\gamma+4\tau)}}\,Y^d(p_\alpha) - \sum_{l=1}^{l^*}\varepsilon_l\,\frac{|\sigma_2|\,E(\sigma_2 z)}{\sqrt{\gamma(\gamma - 4\tau)}}\,Y^d(|\sigma_2|)\right\} \tag{1.8}$$

when $s_c^* z \to +\infty$

$$G_s^c = \frac{bi}{4\pi}\int\limits_{\pi/2}^{\pi-\gamma_0} I^-\,d\lambda \tag{1.9}$$

$$G_s^d = \frac{bi}{4\pi}\sum_{l=1}^{l^*}\varepsilon_l\,\frac{|\sigma_1|\,E(\sigma_1 z)}{\sqrt{\gamma(\gamma-4\tau)}}\,Y^d(|\sigma_1|) \quad \text{when } s_c^* z \to -\infty \tag{1.10}$$

Here

$$I^\pm = \frac{s_c^\pm E(s_c^\pm z\cos\lambda)}{\sqrt{1 + 4\tau\cos\lambda}}\,Y^c$$

$$s_c^\pm = s_c^*\,\frac{1 + 2\tau\cos\lambda \pm \sqrt{1 + 4\tau\cos\lambda}}{2(1+\tau^2)\cos^2\lambda}, \quad s_c^* = \nu\,\frac{1+\tau^2}{\tau^2}$$

We shall use the stationary-phase method /9/ for the subsequent estimation of the integrals in (1.7) and (1.9). The value of the integral

$$J = \int\limits_a^b I^\pm(\lambda)\,E(W^* S(\lambda))\,d\lambda$$

at large values of the parameter $W^* = s_c^* R$, where $R = \sqrt{(r - r_0)^2 + z^2}$, is determined by the expression

$$J \simeq \sum_{\{\lambda_0\}} \sqrt{\frac{2\pi}{W^* |S''(\lambda_0)|}}\, I^{\pm}(\lambda_0) E\left(W^* S(\lambda_0) + \operatorname{sgn} S''(\lambda_0) \frac{\pi}{4}\right)$$

where λ_0 are the roots of the equation $S'(\lambda_0) = 0$. Hence, the subsequent estimation involves a search for the roots of the following equations:

$$\frac{ds_c^{\pm}}{d\lambda} = s_c^{\pm} \frac{z \sin \lambda + M \cos \lambda}{z \cos \lambda - M \sin \lambda} \tag{1.11}$$
$$M = (-1)^{\chi} r \cos (a_{kj} + (-1)^j \theta) + (-1)^{\chi'} r_0 \cos (a_{k'j'} + (-1)^{j'} \theta_0)$$

In order to determine the roots of these equations we shall confine ourselves to the case of small values of τ which satisfy the condition $\tau < 1/4$. When $\nu z \to \infty$, expressions (1.7)-(1.10) can be represented, apart from terms containing τ^2, in the form

$$G_s^c = \frac{b\nu i}{4\pi} \int_0^{\pi/2} (1 - 4\tau \cos \lambda) E(\nu z (1 - 2\tau \cos \lambda) \cos \lambda)\, Y^c\, d\lambda \tag{1.12}$$

$$G_s^d = \frac{b\nu i}{4\pi} \sum_{l=1}^{n-1} \varepsilon_l \frac{1}{\gamma^2}\left(1 - \frac{4\tau}{\gamma}\right) E\left(\frac{\nu |z|}{\gamma}\left(1 - \frac{2\tau}{\gamma}\right)\right) Y^d\left(\frac{\nu}{\gamma}\left(1 - \frac{2\tau}{\gamma}\right)\right) \tag{1.13}$$

and, when $\nu z \to -\infty$, the quantity G_s^c is determined according to a formula which differs from (1.12) in that the integral is taken from $\pi/2$ to π and G_s^d is determined using a formula which differs from (1.13) in that τ is replaced by $-\tau$.

The asymptotic estimate of G_s^c has the form

$$G_s^c \simeq \frac{bi}{2\pi}\left(\frac{\nu}{2\pi R}\right)^{1/2} D\left(D\left(I \frac{1 - 4\tau \cos \lambda_0}{|S''(\lambda_0)|^{1/2}} \exp(-\nu (1 - 2\tau \cos \lambda_0) \times \right.\right.$$
$$\left.\left.(\bar{M} - i(z \cos \lambda_0 - M \sin \lambda_0)) + \frac{\pi i}{4} \operatorname{sgn} S''(\lambda_0)\right)\right)\right)$$
$$I = \exp\left\{i \frac{\pi}{4}(n-1)[(-1)^{\chi} + (-1)^{\chi'}]\right\}, \quad D = \frac{1}{2\sqrt{2\pi}} \sum_{j,\chi=0}^{1} \sum_{k=1}^{n} B_k^{(\chi)}$$
$$S''(\lambda_0) = z(4\tau \cos 2\lambda_0 - \cos \lambda_0) - M(4\tau \sin 2\lambda_0 - \sin \lambda_0)$$

where λ_0 is a root of the equation

$$z(2\tau \sin 2\lambda - \sin \lambda) + M(2\tau \cos 2\lambda - \cos \lambda) = 0$$

$\lambda \in [0, \pi/2]$ when $\nu z \to +\infty$, $\lambda \in [\pi/2, \pi]$ when $\nu z \to -\infty$ and $\bar{M}$ is calculated using a formula which differs from (1.11) by the condition that $\chi = \chi' = 0$ and by the replacement of cos by sin.

Analysis of the resulting solution enables us to investigate the structure of the marine waves which are a particular realization of Cherenkov radiation in a dispersive medium. The radiation field is formed by the waves of the continuous and discrete spectra and cosists of two characteristic zones: the neighbourhood of the source and the coastal zone. In the neighbourhood of the source the wave motion is provided by the waves of the continuous spectrum and consists of a wave field which is symmetrical about the $z = 0$ axis and is caused by the pulsation of the source and of the wave wake which is localized in the tail part and is caused by the translational motion of the source. The waves of the discrete spectrum provide the motion in the coastal zone. A Stokes wave is found in the composition of the waves of the discrete spectrum. This wave is greater than all the remaining waves in the neighbourhood of the shore line and it may be assumed that the wave motion which propagates along the shore line is solely formed by the Stokes wave.

The asymptotic estimates which have been obtained show that, for small τ, a moving and pulsating source radiates waves which diverge on all sides from the source. These waves, unlike the radiation from a fixed pulsating source, depend on direction, and a complex radiation pattern is formed. Cherenkov radiation (marine waves) is superimposed on this main wave field. If the emission frequency ω is low and g/c^2 is a finite quantity, then the Cherenkov radiation with characteristic features which are inherent when there is a sloping bottom predominates. $G_s \to 0$ when $\tau \gg 1/4$.

2. Let us now consider the special cases of a moving source of constant intensity ($\omega = 0$) and a fixed pulsating source ($c = 0$) which allow the problem to be solved in explicit form.

Let $\omega = 0$. In this case the parameters of the problem are determined using the formulae

$$s_c^+ = \frac{g}{c^2 \cos^2 \lambda}, \quad p_1 = -\sigma_2 = \frac{g\gamma}{c^2}, \quad s_c^- = p_2 = \sigma_1 = \gamma_0 = 0, \quad l^* = n-1$$

and the functions G^c and G^d, obtained from formulae (1.4) and (1.5) using Cauchy's residue theorem, have the form

$$G^c = \frac{b}{4\pi} \int\limits_{g/c^2}^{\infty} \sin\left(\sqrt{\frac{gs_c^+}{c^2}}\, z\right) Y^c \left(\frac{c^2 s_c^+}{g} - 1\right)^{-1/2} ds_c^+ \tag{2.1}$$

$$G^d = \frac{bg}{8\pi c^2} \sum_{l=1}^{n-1} \varepsilon_l \sin\left(\frac{g\gamma}{c^2} \cdot z\right) Y^d \left(\frac{g\gamma}{c^2}\right) \tag{2.2}$$

When constructing formulae (2.1) and (2.2) which describe the zone of Cherenkov emission, the condition that there is no wave front in front of the source we ensured. On the basis of (2.1) and (2.2) we calculate the elevation of the free surface

$$\eta = \eta^c + \eta^d \tag{2.3}$$

$$\eta^c = \frac{2bg}{c^3} \sum_{\alpha=0}^{1} D\Big(D\Big(I \int\limits_0^{\infty} \sqrt{1+u^2} \exp\Big(-\frac{g}{c^2}\sqrt{1+u^2} \times$$

$$(\sqrt{1+u^2}\, \overline{M}_0 + i\,((-1)^\alpha z + M_0 u))\Big) du\Big)\Big)$$

$$M_0, \overline{M}_0 = M, \quad \overline{M}\ (\theta = 0)$$

$$\eta^d = \frac{bg}{2\pi c^3} \sum_{l=1}^{n-1} \varepsilon_l \gamma \cos\left(\frac{g\gamma}{c^2}\, z\right) Y^d \left(\frac{g\gamma}{c^2}\right)\Big|_{\theta=0} \tag{2.4}$$

Let us now carry out an asymptotic analysis of expression (2.3) using the stationary-phase method (the large parameter $W = gR/c^2$). For this purpose we study the behaviour of the phase function

$$S(u) = R^{-1} (1 + u^2)^{1/2} ((-1)^\alpha z + M_0 u)$$

The stationary points $S(u)$ are calculated using the formula

$$u_\pm = [-(-1)^\alpha z \pm \sqrt{z^2 - 8M_0^2}]/(4M_0)$$

The condition that the roots $u_\pm$ are real and positive determines the domain of the existence of the wave motion of the fluid caused by the motion of the source:

$$0 \leqslant M_0 \leqslant z/(2\sqrt{2})$$

At the points of inflection of the phase function $(S''(u_\pm) = 0)$, the stationary points $u_\pm = 2^{-1/2}$ are degenerate, which is allowed for in writing the asymptotic formulae. Hence:

$$\eta^c \simeq \frac{2b}{c^2} \left(\frac{2\pi g}{R}\right)^{1/2} \sum_{\{u_\pm\}} \sum_{\alpha=0}^{1} D^\alpha \Big(D\Big(I \left(\frac{(1+u_\pm^2)^3}{z^2 - 8M_0^2}\right)^{1/4} \times$$

$$\exp\Big(-\frac{g}{c^2}\sqrt{1+u_\pm^2}\Big(\sqrt{1+u_\pm^2}\, \overline{M}_0 + i\,((-1)^\alpha z + M_0 u_\pm)) \mp i\,\frac{\pi}{4}\Big)\Big)\Big)$$

where $0 \leqslant M_0 < z/(2\sqrt{2})$

$$\eta^c \simeq \left(\frac{2}{3}\right)^{1/2} \frac{b}{c^2} \Gamma\left(\frac{1}{3}\right) \left(\frac{g^2}{cR}\right)^{1/3} \sum_{\alpha=0}^{1} D^\alpha \Big(D\Big(I \times$$

$$\exp\Big(-\frac{3g}{2c^2}\, \overline{M}_0 + \frac{3\sqrt{3}g}{2c^2} M_0 i - (-1)^\alpha \frac{\pi}{6}\, i\Big)\Big)\Big)$$

where $M_0 = z/(2\sqrt{2})$.

D^0 and D^1 are conditional operators: D^0 indicates that the summation is carried out over indices which satisfy the condition $-(-1)^\varkappa \cos a_{kj} > 0$ and D^1 indicates that summation is carried out over indices which satisfy the condition $(-1)^\varkappa \cos a_{kj} > 0$.

Let us now analyse expression (2.4). In the coastal zone the magnitude of η^d is calculated using (2.4) in which $r \simeq 0$. In the neighbourhood of the shore line, the Stokes wave is the decisive wave of the waves of the discrete spectrum and the wave motion corresponding to it has the form

$$\eta^d_{n-1} \simeq \frac{bg}{2\pi c^3}\,\varepsilon_{n-1}\sin\beta\cos\left(\frac{g\sin\beta}{c^2}\,z\right)Y^d\left(\frac{g\sin\beta}{c^2}\right)\Big|_{l=n-1,\ r=\theta=0} \tag{2.5}$$

As can be seen from (2.5), as the source becomes farther from the shore line, the waves of the discrete spectrum decay exponentially.

Let us describe the structure of the wave field which is formed during the motion of the source over the surface of a fluid of variable depth $(\theta_0 = 0)$. Unlike the case of an infinitely deep fluid in which the Cherenkov radiation is concentrated in a wedge with an aperture angle of 19°28′ behind the moving source, in the case of an inclined bottom it forms a domain bounded on one side from the line of motion of the source to the shore line $r = 0$, along which the Stokes wave propagates and, on the other side, when $r > r_0$, by the line

$$r = z\,(2\sqrt{2})\cos(\pi m/n))^{-1} + r_0\,(\cos(\pi m/n))^{-1} \tag{2.6}$$

We shall call the line which is described by Eq.(2.6) the boundary of the wave field. As the angle of inclination of the bottom β decreases, the boundary of the wave field unfolds to an ever greater angle with respect to the line of motion of the source and, when $\beta \to 0$, the line (2.6) subtends an angle with the line of motion which is close to $\pi/2$.

As the source becomes more remote from the shore, the shore line ceases to play the role of a boundary of the domain of radiation when $r < r_0$, and the straight line

$$r = r_0\cos(\pi m/n) - z/(2\sqrt{2})$$

becomes the boundary of the domain.

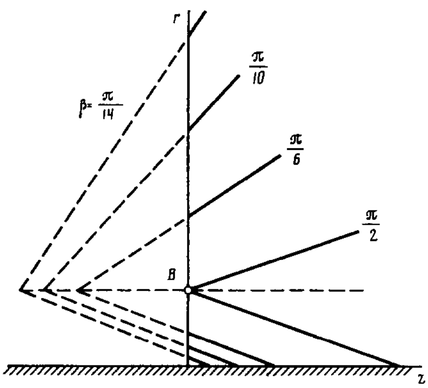

Fig.3

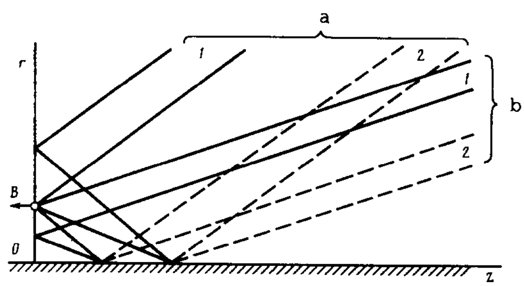

Fig.4

The aperture angle of the Cherenkov wedge increases and, at the same time, the wedge is unfolded with respect to the line of motion of the source by the vertex to the shore (Fig.3).

The internal filling of the Cherenkov wedge is a result of the superpositioning of longitudinal and transverse waves. Waves possessing a rectilinear front, the equation of which has the form $M = z/2\sqrt{2}$, and which are characterized by the fact that there is a significant increase in the amplitudes of the wave field (of the longitudinal and transverse waves) in these lines are formed on the background of such wave formations which decay as $R^{-1/2}$. We shall refer to the above-mentioned lines as divergent waves. Such waves also exist in the case of an infinitely deep fluid which is unbounded in the plane and are the boundaries of the wave wake. The system of divergent waves consists of $m+1$ families and each family of waves is characterized by the one and the same aperture angle $(|\cos a_{kj}| = \cos(\pi s/n),\ s = 0, 1, \ldots, m)$ and includes $m+1$ direct waves and $m+1$ waves which have been reflected from the shore.

In the domain $r > r_0$, the direct waves depart to infinity, decaying as $R^{-1/2}$. In the domain $r < r_0$, the $m+1$ direct waves are incident upon the shore at certain points with the coordinates $z = 2\sqrt{2}r_0\cos(\pi s/n)$, $s = 0, 1, \ldots, m$ (we shall refer to these as shore points) and subsequently form reflected waves. A fan consisting of $m+1$ direct waves (one from each family) arrives at each shore point and a fan of reflected waves, of which there are also $m+1$. leaves the shore. The direct and reflected waves are symmetrically arranged with respect to the lines

$$r = z\,(2\sqrt{2}\cos(\pi s/n))^{-1} \tag{2.7}$$

in a band with a width of $2r_0(\cos(\pi s/n))^{-1}$ (s is the number of the family). The axes of symmetry of each of the $m+1$ families are divergent waves from the source which move along the shore line ($r_0 = 0$). In this case the reflected waves merge with the direct waves and each s-th family degenerates into a single line (2.7). The family of direct and reflected waves closest to the shore ($s = 0$) is inclined at an angle of $19°28'$ for any β which corresponds to the motion of a source in an infinitely deep fluid.

The structure of the wave wake when $\beta = \pi/6$ ($m = 1$) is shown in Fig.4 where the numbers 1 and 2 indicate the direct and reflected waves respectively and the letters a and b refer to the numbers of the wave families which are characterized by the same aperture angle.

One can be guided by the following rule in constructing the wave pattern: a system consisting of $2m+1$ fictitious sources, located on the $z = 0$ axis and removed from the shore line to a distance of

$$r = \pm r_0/\cos(\pi s/n), \quad s = 0, 1, \ldots, m$$

is added to the real source which is located on the $z = 0$ axis at a distance r_0 from the shore line.

A ray at an angle of $\operatorname{arctg}[(2\sqrt{2}\cos(\pi s/n))^{-1}]$ passes from each of the $2m+2$ sources and, moreover, two fictitious sources, arranged symmetrically about the $r = 0$ axis, correspond to each s. At large distances from the source, the rays leaving the $m+1$ sources, located on the positive part of the r axis, correspond to direct waves while the rays leaving the $m+1$ sources located on the negative part of the r axis correspond to waves which have been reflected from the shore.

The problem of the structure of the wave wake which is formed during the motion of a source over a planar inclined bottom was considered for the first time in /5/ for values of the angles $\beta = \pi/4$ and $\pi/6$. However, the mixed nature of the eigenfunction spectrum of the problem was not established and, as a result of this, only one component of the solution was found which is associated with the eigenfunctions of the continuous spectrum. Only a single wave was found from the combination of the divergent waves which corresponds to the wave which is formed when a source moves in a fluid half-space. A qualitative interpretation of the solution /5/ is given in the first edition (published in 1936) of /1/ (pp.117-118). However, the wave picture which is presented is incomplete in view of the above-mentioned constraint on the solution in /5/.

Comparison of the quantities η^c and η^d shows that the wave motion due to the waves of the continuous spectrum decays close to the shore line as $z^{-1/2}$, while the motion due to the waves of the discrete spectrum involves the propagation of non-decaying waves.

On the basis of the results which have been described, it is possible to obtain a formula for the wave resistance F of a body of the Mitchell type which moves at a constant velocity c parallel to the shore line at a distance x_0 from it

$$F = \frac{2\rho g^2}{\pi c^2}\left\{\int_1^\infty \frac{I_c^2 + J_c^2}{\sqrt{\lambda^2 - 1}}\lambda^2 d\lambda + \sum_{l=1}^{n-1}\varepsilon_l \gamma (I_d^2 + J_d^2)\right\} \tag{2.8}$$

$$\left.\begin{matrix} I_c \\ J_c \end{matrix}\right\} = \iint_{(S)} f_z \Phi^c(x_0, y_0) \begin{Bmatrix}\cos\\ \sin\end{Bmatrix}\left(\frac{g}{c^2} z_0 \lambda\right) dy_0\, dz_0$$

$$\left.\begin{matrix} I_d \\ J_d \end{matrix}\right\} = \iint_{(S)} f_z \Phi_l^d(x_0, y_0) \begin{Bmatrix}\cos\\ \sin\end{Bmatrix}\left(\frac{g}{c^2} z_0 \gamma\right) dy_0\, dz_0$$

where $|x - x_0| = f(y, z)$ is the equation of the contour of the body, y_0, z_0 are the coordinates of points on the surface of the body S around which the flow occurs and ρ is the density of the fluid.

The dependence (2.8) is a generalization of Mitchell's formula in a domain of variable depth /1/.

Let $c = 0$. When $\tau = 0$, in the case of the pulsation of a fixed source, we get from (1.12) and (1.13) that

$$\eta = i\omega g^{-1} E(-\omega t)\, G_s |_{\theta=0} \tag{2.9}$$

$$G_s^c = \frac{bi}{2}\sqrt{\frac{\nu}{2\pi}}\sum_{\alpha=0}^{1} D^\alpha (D(I(z^2 + M^2)^{-1/4}\exp(-\nu\bar{M} + i(-1)^\alpha \times$$

$$(\nu\sqrt{z^2 + M^2} - \pi/4))))$$

$$G_s^d = \frac{bi\nu}{2\pi}\sum_{l=1}^{n-1}\varepsilon_l \gamma^{-2}\cos\left(\frac{\nu}{\gamma} z\right) Y^d\left(\frac{\nu}{\gamma}\right)$$

In the coastal zone, the wave field is determined using the formula and by putting $G_s^c = O(z^{-1/2})$ in the last formula of (2.9) when $r = 0$.

In the neighbourhood of the shore line, the Stokes wave is the decisive wave of all the waves of the discrete spectrum and the wave motion corresponding to it has the form

$$G^d_{n-1} \simeq \frac{biv}{2\pi}\varepsilon_{n-1}\sin^{-2}\beta\cos\left(\frac{v}{\sin\beta}z\right)Y^d\left(\frac{v}{\sin\beta}\right)\Big|_{r=0,\ l=n-1}$$

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Translated by E.L.S.

J. Appl. Maths Mechs, Vol. 55, No. 3, pp. 332-337, 1991
Printed in Great Britain

0021-8928/91 $15.00+0.00

ON THE RATE OF PROPAGATION OF SMALL PERTURBATIONS IN POROUS MEDIA*

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The properties of a system of equations which describes the two-speed motion of a porous medium are investigated. The type of system of equations is defined as a function of the rate of slippage of the phases and the difference in the stresses in the phases. The domains of variation of the decisive parameters for which the system of equations describing the dynamics of a two-phase porous medium remains hyperbolic are established.

For the correct formulation of the problem of the two-speed flow of a compressible porous medium it is necessary to determine the type of corresponding system of differntial equations. There are a considerable number of papers dealing with similar kinds of investigations for various systems of equations describing the motion of multiphase media. The equations of continuity and the equations of motion are written out for each phase: an assumption concerning barotropicity is used for the closure of the system and the non-hyperbolic nature of such a system of equations is indicated for real values of the difference in the speeds of the phases /1, 2/. It has been shown /3/ that, in the more general case for the complete system of equations which describes the flow of compressible phases using a model containing the same pressure for the different phases, the system of differential equations is not hyperbolic for real values of the magnitude of slippage. The propagation of small perturbations in a mixture with a barotropic gas phase has been investigated: it was noted that the non-hyperbolic character and instability of the small perturbations which are typical of the system of differential equations are attributable to an insufficiently complete description of the interphase interactions within the disperse

**Prikl.Matem.Mekhan.*,55,3,410-415,1991